The power of the handshaking lemma

Daniel Paleka

July 2021

§1 Preliminaries

All graphs are *simple*: undirected, no loops, no multiple edges.

Definition 1.1. We call a connected graph *Eulerian* if all its vertices have even degree. This is the necessary and sufficient condition for the graph to have an *Euler tour*: a walk visiting each edge exactly once and returning to the starting vertex.

Lemma 1.2 (Handshaking)
Any graph has an even number of vertices with odd degree. Furthermore, any connected component has an even number of vertices with odd degree.

§2 Problems

Harder problems are marked by asterisks.

1. (Sperner's lemma) Let a *triangulation* of a triangle be a partition of its area into triangles with disjoint interiors. The vertices of the triangles in the partition are the *vertices of the triangulation*.

   Given a triangle ABC, and a triangulation T of the triangle, the set S of vertices of T is colored with three colors in such a way that

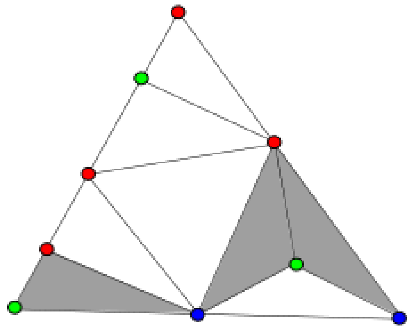
   - A, B, and C are colored 1, 2, and 3 respectively
   - Each vertex on an edge of ABC is to be colored only with one of the two colors of the ends of its edge. For example, each vertex on AC must have a color either 1 or 3.

   Prove that there exists a triangle from T, whose vertices are colored with three different colors.

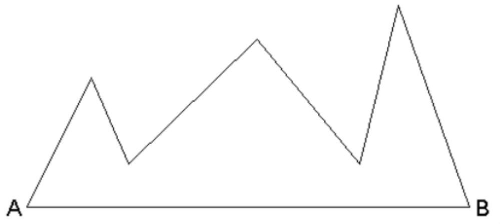
2. (Mountain climbing problem) A *mountain range* is a piecewise linear continuous function f defined on $[0, 1]$ which satisfies $f(0) = f(1) = 0$ and $f(x) > 0$ for all $0 < x < 1$. It can be shown that f has finitely many pieces.

   There is a hiker A at the point $(0, 0)$ and a hiker B at the point $(1, 0)$. The two hikers begin moving along the mountain range, the only restriction being that they must always stay at the same y-coordinate (height).

   Prove that A and B can always meet at some point of the mountain range.



a) A triangulation from Problem 1. Triangles with three colors are marked.



b) An example of a mountain range.

3. Let G be an Eulerian graph on an even number of vertices. Prove that G has an even number of spanning trees.

$*$4. Let G be an Eulerian graph. Prove that each edge is in an odd number of cycles.

5. Let G be a graph with all vertices of odd degree. Prove that each edge is in an even number of Hamiltonian cycles.

§3 Additional problems

$*$6. Let $G = (V, E)$ be a graph. Call a subset of vertices A *dominating* if any vertex $v \in V \backslash A$ has a neighbour in A. Prove that G has an odd number of dominating sets.

$*$7. Let G be a graph, and let x and y be two vertices of G. Suppose that all vertices except x and y have odd degree. Prove that the number of Hamiltonian paths from x to y is even.

$**$8. (Christofides algorithm) [1] Let (V, d) be a finite metric space: a weighted symmetric complete graph on V with edge weights $d(u, v)$ for $u, v \in V$ such that the triangle inequality holds: $d(u, v) \leq d(u, x) + d(x, v)$ for all $u, v, x \in V$. Let C be the cost of the Hamiltonian cycle with the minimum total cost.

Give an algorithm that finds a Hamiltonian cycle with cost at most $\frac{3}{2}C$ in polynomial time.

(This shows that the so-called metric TSP is $\frac{3}{2}$-*approximable.*)

[1] This is only tangentially related to the topic, but it could be nice to think about if this lecture was too easy for you.

§4 Hints/Solutions

The hints give the main idea of the solution, with the details left to fill out. Everything is passed through ROT13. [2]

1. Gur iregvprf bs gur tencu ner gur gevnatyrf naq gur bhgfvqr ertvba. Qrsvar gur rqtrf hfvat gur pbybef orgjrra gjb ertvbaf.
2. Gur iregvprf bs gur tencu ner gur fgngrf jura uvxref ner ng gur fnzr urvtug. Bayl n svavgr fhofrg bs gur hapbhagnoyr frg bs fgngrf vf eryrinag sbe gur ceboyrz.
3. Gur iregvprf ner gur fcnaavat gerrf, pbaarpg gurz jura gurl qvssre ol bar rqtr.
4. Hint 1: cebir gung jr pna ercynpr "plpyrf" ol "pvephvgf" va gur ceboyrz.

   Hint 2: Gur iregvprf ner nyy aba-rqtr-ercrngvat cnguf fgnegvat jvgu gur pubfra rqtr va n svkrq qverpgvba. Gjb cnguf ner nqwnprag vss lbh pna qryrgr gur ynfg rqtr bs bar gb bognva gur bgure.
5. Gur iregvprf ner Unzvygbavna cnguf fgnegvat h-i. Gjb cnguf ner nqwnprag jura lbh pna bognva bar sebz gur bgure ol nqqvat n onpx-rqtr ng gur raq naq erzbivat gur erqhaqnag rqtr va gur zvqqyr.

§5 Sources

1. Wikipedia: Sperner's lemma
2. Wikipedia: Mountain climbing problem
3. Math StackExchange: Equivalent characterization of eulerian circuits
   MathOverflow: Parity results via the handshaking lemma
4. Mathlinks: Eulerian graph
5. MathOverflow: Parity results via the handshaking lemma
6. MathOverflow: Parity results via the handshaking lemma
7. MathOverflow: Handshaking lemma
8. Wikipedia: Christofides algorithm

[2] This is intended to be a nuisance, so that the reader reconsiders looking up the solution. Sometimes you need to try just a bit harder.