Algorithms and posets

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All graphs, unless stated otherwise, are *simple*: undirected, no loops, no multi-edges. Harder problems are marked by asterisks.

§1 Simple algorithms on graphs

1. There are n students in a class. Some of them are friends of each other. (Friendship is mutual.) No one is friends with more than k other students. Prove that each student can choose one of $k+1$ colors such that no two friends are the same color.

2. An epidemic is raging on a summer camp. Each person can be either *healthy*, *sick*, or *recovering*. The first day, some people get sick, while the rest are healthy. On each day, healthy friends of people who were sick on the previous day get sick; people who were sick on the previous day become recovering; and the people who were recovering on the previous day become healthy. Prove that the epidemic will eventually stop.

3. (IMO 1991) Suppose G is a connected graph with k edges. Prove that it is possible to label the edges $1, 2, \ldots, k$ in such a way that at each vertex which belongs to two or more edges, the greatest common divisor of the integers labeling those edges is equal to 1.

4. Let T be a tree with t edges. Prove that every nonempty graph $G = (V, E)$ with average degree at least $2t$ contains T as a subgraph.

§2 Chains and antichains

In what follows, please remember to **always have an example in mind** when saying formal statements.

Definition 2.1. We call a pair of a set S and a binary relation $\lesssim$ on S a *partially ordered set* (or *poset*) if:

- $x \lesssim x$ for all $x \in S$, but for distinct $x, y \in S$, it can't be both $x \lesssim y$ and $y \lesssim x$.
- $x \lesssim y$ and $y \lesssim z$ implies $x \lesssim z$,
- there are no cycles (except loops) in the directed graph with vertices S and edges $x \to y$ whenever $x \lesssim y$.

Example 2.2

The set of positive integers $\mathbb{N}$ is a poset with the usual order $\leq$. It is also a poset with the order $|$, where $a \mid b$ means that a divides b.

Example 2.3

Given a set S, the set $\mathcal{P}(S)$ of all subsets of S is a poset with the order $\subseteq$.

Question 2.4. Give two more examples of posets that are not very similar to the above examples. Hint for one of those: do something with $\mathbb{Z}^2$.

Definition 2.5. A **chain** in a poset $(S, \lesssim)$ is a subset $C \subseteq S$ such that for any $x, y \in C$, either $x \lesssim y$ or $y \lesssim x$.

Example 2.6

Given the poset of all subsets of $\{1, 2, 3, 4, 5\}$ with the order $\subseteq$, the set $\{\emptyset, \{4\}, \{1, 3, 4\}, \{1, 3, 4, 5\}\}$ is a chain.

Question 2.7. Give a very different example of a chain in a poset.

Definition 2.8. An **antichain** in a poset $(S, \lesssim)$ is a set $A \subseteq S$ such that for every pair $x \neq y \in A$, neither $x \lesssim y$ nor $y \lesssim x$.

Example 2.9

Given the poset $\mathcal{P}([n])$ of all subsets of $[n] = \{1, 2, \ldots, n\}$ with the order $\subseteq$, the set

$$\binom{[n]}{2} = \{\{x, y\} \mid x, y \in [n], x \neq y\}$$

of size $\binom{n}{2}$ is an antichain.

Question 2.10. Give a very different example of an antichain in a poset.

In a poset, a chain and an antichain intersect in at most one element.

Question 2.11. Convince yourself of the above statement, both on examples and formally. You should not proceed further until you find this completely obvious.

Theorem 2.12 (Dilworth's Theorem)

In any finite poset, the maximum size of an antichain is equal to the minimum number of chains needed to cover the poset.

Theorem 2.13 (Mirsky's Theorem)

In any finite poset, the maximum size of a chain is equal to the minimum number of antichains needed to cover the set.

Question 2.14. Check that both of these statements are true for all of your examples. The number in Dilworth's Theorem is called the *width* of the poset. Find the width of all your examples of finite posets.

Question 2.15. Prove that the maximum size of an antichain is *less than or equal to* the minimum number of chains needed to cover the poset. Prove the analogous statement for the maximum size of a chain. Please do not proceed further until you find this easy.

Theorem 2.16 (Lubell-Yamamoto-Meshalkin)

Let A be a family of subsets of $[n]$ such that no set in A is a subset of another set in A, and let a_k denote the number of sets of size k in A. Then

$$\sum_{k=0}^{n} \frac{a_k}{\binom{n}{k}} \leq 1.$$

Proof. This is an introductory example for the *probabilistic method* in combinatorics. The non-probabilistic proof uses the following fact: given a subset S of $[n]$ with $|S| = k$, there are $k!(n-k)!$ permutations of $[n]$ that have the elements of S for their first k elements. We leave the rest of the proof as an exercise.

Problems

5. Finish the details of the proof of the Lubell-Yamamoto-Meshalkin inequality.

6. (Sperner's theorem) Let U be an n-element set, let A be a family of subsets of U such that no set in A is a subset of another set in A. Then $|A| \leq \binom{n}{\lceil n/2 \rceil}$.

$*$7. (CEOI 2014) Let N and H be positive integers, with H even. There is a System that has N questions (for simplicity, encoded as integers 1 to N) that can be answered with either "yes" or "no". It is known that the System will ask either question x or question y ($1 \leq x \neq y \leq N$), where the correct answer to x is "yes" and the correct answer to y is "no". Ana and Banana, however, do not know the pair (x, y) a priori, nor they know the correct answer to any question.

   The protocol is as follows:

   a) Ana learns the pair (x, y).

   b) Ana communicates an integer h ($1 \leq h \leq H$) to Banana.

   c) System asks the question $q \in \{x, y\}$.

   d) Banana answers either "yes" or "no".

   Ana and Banana are allowed to devise a strategy beforehand. Given H, what is the largest possible value of N for which Banana can always answer correctly?

8. (COCI 2020/21) Consider a directed *divisibility graph* where $V \subseteq \{1, 2, \dots, 2024^5\}$, $|V| = 2024$, and $E = \{(u, v) : u \in V, v \in V \backslash \{u\}, u|v\}$. Prove that we can color the *edges* of the graph in three colors such that there is no monochromatic path of three edges.

Hint for the following two problems: when you write *poset*, think $\mathcal{P}([n])$ and $\subseteq$.

9. Prove Mirsky's Theorem.

*10. Prove Dilworth's Theorem.

§3 Hints/Solutions

The hints give the main idea of the solution, with the details left to fill out. Everything is passed through ROT13. [1]

1. Whfg qb vg. Bgurejvfr, qrcgu svefg frnepu.

2. Oernqgu svefg frnepu.

3. Qrcgu svefg frnepu. Trarenyvmr gur rnfvrfg fbyhgvba sbe n cngu.

4. Hint 1: Cebir gung rirel tencu jvgu pbagnvaf na vaqhprq fhotencu jvgu zvavzny qrterr unys gur nirentr qrterr bs gur bevtvany tencu.

   Hint 2: Abj vg vf fvzvyne gb gur svefg ceboyrz va guvf unaqbhg.

7. Hint 1: Gur xrl fgrc vf gb erqhpr bar bs gur obhaqf ba A gb gur cerivbhf ceboyrz.

   Hint 2: Guvax bs fbzrguvat fvzvyne gb bhe nagvpunva rknzcyr.

8. Hint 1: Sbe n gung qvivqrf o, jung pna jr fnl nobhg gur ovanel qvtvgf bs n naq o?

   Hint 2: Gur tencu lbh arrq gb fbyir gur ceboyrz ba vf pbzcyrgr qverpgrq nplpyvp tencu jvgu ybt znk ahzore (vf yrff guna fvkgl sbhe) iregvprf.

[1] This is intended to be a nuisance, so that the reader can reconsider instead of looking up the solution. Sometimes, a little more effort is all that's needed.

§4 Sources and solutions

3. https://artofproblemsolving.com/community/c6h60727p366446
4. https://www.math.cmu.edu/~ploh/docs/math/mop2009/graph-theory-extremal.pdf
6. https://en.wikipedia.org/w/index.php?title=Sperner%27s_theorem&oldid=1110420126
7. https://usaco.guide/problems/ceoi-/solution2014question
8. https://hsin.hr/coci/archive/2020_2021/contest4_tasks.pdf → Hop
   The idea is: if $a \mid b$ and $a < b$, then $a \leq 2b$. Let $\text{greatest\_bit}(x)$ equal the position of the leading bit in the binary representation of x. We give the edge between a and b to color 1 if $\lfloor \text{greatest\_bit}(a) \rfloor = \lfloor \text{greatest\_bit}(b) \rfloor$. All other edges we give to color 2 if $\lfloor \text{greatest\_bit}(a) \rfloor = \lfloor \text{greatest\_bit}(b) \rfloor$, and the remaining ones we give to color 3. Using $\text{greatest\_bit}(x) < 64$, we can prove this construction works.